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On the analytical summation of Fourier series and its relation to the asymptotic behaviour of Fourier transforms

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Abstract. Forms of the Poisson summation formula (PSF) appropriate for the summation of semi-infinite and infinite Fourier series are derived. Application of these results to the acceleration of convergence of various types of series with monotonically decreasing coefficient functions yields transformed series with terms that decay either exponentially or with the inverse first or second power of the index variable. These two very different convergence properties are explained in terms of the asymptotic properties of the relevant Fourier transforms, which are in turn related to the power series expansions of the summand functions in the original Fourier series. The result is that the Poisson summation formula works best for Fourier cosine series in which the summand functions are expansible in even powers, and for Fourier sine series in which the summand functions have power series with odd powers. Here, application of the PSF produces series of terms that decay exponentially with increasing argument x. In contrast, application of the semi-infinite version of the PSF to Fourier cosine series of terms with odd-power expansions, or to Fourier sine series of terms with even-power expansions yields transformed series involving functions of the form $\exp(x)\mathrm{E}_1(x) \pm \exp(-x)\,\mathrm{Ei}(x)$, which decay approximately as $1/x$. If the summand function in the Fourier series has a power series with both even and odd powers, the transformed series involves sine and cosine integral functions, which decay approximately as $1/x^2$. Fourier series of these last three types in general require additional acceleration, for example, by application of the Kummer transformation.

1. Introduction

Fourier series arise in many branches of mathematical physics involving the solution of linear second-order partial differential equations. In most cases, however, the slowness of convergence of such series precludes their usefulness in numerical calculations. In recent work [1] it was shown how the Poisson summation formula (PSF) could be applied to the acceleration of Fourier-series of the Laplace equation. This success suggests that it might be desirable to examine the application of the PSF to Fourier series of more general kinds, such as that considered in a recent paper by Oleksy [2]. From the form of the PSF, it is clear that the extent of acceleration so obtained is determined by the asymptotic behaviour of the Fourier transforms of the functions in the original Fourier series. The asymptotic behaviour of a transform for large arguments is usually considered in relation to the behaviour of the original function for small arguments, and *vice versa*. While many such Tauberian theorems are known for Laplace transforms [3, pp 121–32], surprisingly few results are available for Fourier transforms, and none of these relate the asymptotic behaviour of the Fourier transform of a function to its power series. In the present paper we show how such a relation can be derived, with the aim of providing a rather general way of determining whether the PSF is useful for a given slowly convergent Fourier series. We do this by first considering three specific cases which illustrate the different convergence behaviour of series resulting from application of the PSF.

2. Poisson's formula for semi-infinite series

The Poisson summation formula for an exponential Fourier series

$$\sum_{n=-\infty}^{\infty} f(n)\,\mathrm{e}^{\mathrm{i}cn} = \int_{-\infty}^{\infty} f(u)\,\mathrm{e}^{\mathrm{i}cu}\,\mathrm{d}u + \sum_{m=1}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\,\mathrm{e}^{(2m\pi+c)\mathrm{i}u}\,\mathrm{d}u + \int_{-\infty}^{\infty} f(u)\,\mathrm{e}^{(2m\pi-c)\mathrm{i}u}\,\mathrm{d}u\right] \tag{1}$$

which, for even real-valued functions f reduces to

$$\sum_{n=-\infty}^{\infty} f(n)\,\mathrm{e}^{\mathrm{i}cn} = \int_{-\infty}^{\infty} f(u)\cos cu\,\mathrm{d}u + \sum_{m=1}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\cos(2m\pi + c)u\,\mathrm{d}u + \int_{-\infty}^{\infty} f(u)\cos(2m\pi - c)u\,\mathrm{d}u\right] \tag{1a}$$

can be derived in various ways, as discussed in treatises on complex analysis [4, p 222] and the theory of distributions [5, p 254, 6, pp 46–7]. Its generalization to semi-infinite series is more clearly obtained by a somewhat different route, which also serves to clarify the relation of the PSF to other methods of evaluating series. This approach is suggested by consideration of the series

$$\sum_{n=0}^{\infty}\frac{\mathrm{e}^{2\pi\mathrm{i}nx}}{n+a} \equiv \sum_{n=0}^{\infty}\frac{\cos 2n\pi x}{n+a} + \mathrm{i}\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n+a} \tag{2}$$

where x is between 0 and 1, which arose [2] in a recently proposed model for interaction between atoms in adsorbed monolayers. For present purposes, it is to be observed that an analytical transformation of this series can be constructed by use of the result

$$\sum_{n=0}^{\infty}\frac{\mathrm{e}^{2n\pi\mathrm{i}x}}{n+a} = \int_0^{\infty}\frac{\mathrm{e}^{-az}}{1-\mathrm{e}^{2\pi\mathrm{i}x-z}}\,\mathrm{d}z \tag{3}$$

due to Lerch [7] (see also Whittaker and Watson [8, p 280, problem 8]), which can be readily derived by representing the general term in the exponential form of equation (2) as a Laplace transform integral

$$\frac{\mathrm{e}^{2n\pi\mathrm{i}x}}{n+a} = (\mathrm{e}^{2\pi\mathrm{i}x})^n\int_0^{\infty}\mathrm{e}^{-(n+a)z}\,\mathrm{d}z \tag{4}$$

interchanging the order of summation and integration, and summing the geometric series of exponential functions. This type of integral representation is of considerable importance in the theory of the zeta functions of Riemann and Hurwitz [8, pp 265–80], and is also useful as a method of summing a variety of slowly converging series [9]. It also suggests that an integral representation can be derived for a general Fourier series, in which the coefficient functions $f(n)$ can be represented as a Laplace transform of some other function $G(z)$:

$$\sum_{n=0}^{\infty} f(n)\,\mathrm{e}^{\mathrm{i}cn} = \sum_{n=0}^{\infty}\left\{\int_0^{\infty} G(z)\,\mathrm{e}^{-nz}\,\mathrm{d}z\right\}(\mathrm{e}^{\mathrm{i}c})^n = \int_0^{\infty}\frac{G(z)}{1-\mathrm{e}^{\mathrm{i}c-z}}\,\mathrm{d}z. \tag{5}$$

While this integral could, in principle, be separated into real and imaginary parts and evaluated numerically, it proves more convenient to expand the denominator of the integrand in an infinite series of reciprocal linear terms. This expansion can be obtained by combining the identity

$$\frac{1}{1-\mathrm{e}^{b-z}} = \frac{1}{2}\left[1-\coth\left(\frac{b-z}{2}\right)\right] \tag{6}$$

where b is a constant, with the result of applying the Mittag–Leffler expansion theorem [10, pp 175, 191–2] to the Langevin function

$$L(y) \equiv \coth y - \frac{1}{y} = 2y\sum_{n=1}^{\infty}\frac{1}{y^2+(n\pi)^2} \tag{7}$$

as follows

$$\begin{aligned}\frac{1}{1-\mathrm{e}^{b-z}} &= \frac{1}{2}\left[1-\frac{2}{b-z}-\left\{\coth\left(\frac{b-z}{2}\right)-\frac{2}{b-z}\right\}\right]\\ &= \frac{1}{2}-\frac{1}{b-z}-2(b-z)\sum_{n=1}^{\infty}\frac{1}{(b-z)^2+(2n\pi)^2}\\ &= \frac{1}{2}-\frac{1}{b-z}-\sum_{n=1}^{\infty}\left[\frac{1}{b-z+2n\pi\mathrm{i}}+\frac{1}{b-z-2n\pi\mathrm{i}}\right].\end{aligned} \tag{8}$$

On putting $b = \mathrm{i}c$, there results

$$\frac{1}{1-\mathrm{e}^{\mathrm{i}c-z}} = \frac{1}{2}+\frac{1}{z-\mathrm{i}c}+\sum_{n=1}^{\infty}\left[\frac{1}{z-\mathrm{i}(2n\pi+c)}+\frac{1}{z+\mathrm{i}(2n\pi-c)}\right]. \tag{9}$$

Multiplying by $G(z)$ and integrating from zero to infinity

$$\begin{aligned}\int_0^{\infty}\frac{G(z)}{1+\mathrm{e}^{\mathrm{i}c-z}}\,\mathrm{d}z &= \frac{1}{2}\int_0^{\infty}G(z)\,\mathrm{d}z+\int_0^{\infty}\frac{G(z)}{z-\mathrm{i}c}\,\mathrm{d}z\\ &\quad+\sum_{m=1}^{\infty}\left[\int_0^{\infty}\frac{G(z)}{z-\mathrm{i}(2m\pi+c)}\,\mathrm{d}z+\int_0^{\infty}\frac{G(z)}{z+\mathrm{i}(2m\pi-c)}\,\mathrm{d}z\right].\end{aligned} \tag{10}$$

Now the first of the integrals in the series on the right-hand side of equation (10) can be rearranged as follows

$$\begin{aligned}\int_0^{\infty}\frac{G(z)}{z-\mathrm{i}(2m\pi+c)}\,\mathrm{d}z &= \int_0^{\infty}G(z)\left\{\int_0^{\infty}\mathrm{e}^{-(z-\mathrm{i}(2m\pi+c))t}\,\mathrm{d}t\right\}\mathrm{d}z\\ &= \int_0^{\infty}\mathrm{e}^{(2m\pi+c)\mathrm{i}t}\left\{\int_0^{\infty}G(z)\,\mathrm{e}^{-tz}\,\mathrm{d}z\right\}\mathrm{d}t\\ &= \int_0^{\infty}f(t)\,\mathrm{e}^{(2m\pi+c)\mathrm{i}t}\,\mathrm{d}t\end{aligned} \tag{11}$$

and this result is also valid for $m = 0$. Proceeding similarly for the second integral

$$\int_0^{\infty}\frac{G(z)}{z+\mathrm{i}(2m\pi-c)}\,\mathrm{d}z = \int_0^{\infty}f(t)\,\mathrm{e}^{-(2m\pi+c)\mathrm{i}t}\,\mathrm{d}t. \tag{12}$$

The original Fourier series can, therefore, be written in the form

$$\begin{aligned}\sum_{n=0}^{\infty}f(n)\,\mathrm{e}^{\mathrm{i}cn} &= \frac{1}{2}f(0)+\int_0^{\infty}f(t)\,\mathrm{e}^{\mathrm{i}ct}\,\mathrm{d}t\\ &\quad+\sum_{m=1}^{\infty}\left[\int_0^{\infty}f(t)\,\mathrm{e}^{(2m\pi+c)\mathrm{i}t}\,\mathrm{d}t+\int_0^{\infty}f(t)\,\mathrm{e}^{-(2m\pi-c)\mathrm{i}t}\,\mathrm{d}t\right].\end{aligned} \tag{13}$$

For an even, real-valued function $f(n)$ that is also defined for $n = 0$, the infinite and semi-infinite sums are related by

$$\sum_{n=0}^{\infty} f(n)\cos cn \equiv \frac{1}{2} f(0) + \frac{1}{2} \sum_{n=-\infty}^{\infty} f(n)\,\mathrm{e}^{\mathrm{i}cn} \tag{14}$$

and the infinite and semi-infinite Fourier transforms by

$$\int_0^{\infty} f(u)\cos tu\,\mathrm{d}u = \frac{1}{2}\int_{-\infty}^{\infty} f(u)\cos tu\,\mathrm{d}u = \frac{1}{2}\int_{-\infty}^{\infty} f(u)\,\mathrm{e}^{\mathrm{i}tu}\,\mathrm{d}u. \tag{15}$$

On substitution of equations (14) and (15) into the real part of equation (13), the terms in $f(0)$ cancel, yielding equation (1a).

3. Reciprocal linear coefficients

Oleksy's series [2], for which $c = 2\pi x$ and $G(z) = \exp(-az)$, is evidently an exacting test of any proposed method for convergence acceleration, since the coefficient functions decrease slowly with increasing n, and the series is divergent in the (physically important) limit as x tends to 0. From the practical viewpoint, the series is much more conveniently evaluated by observing that

$$h(z) = \sum_{n=0}^{\infty} \frac{z^n}{n+a} = \frac{1}{a^2} F_1(a, 1, a+1; z) \tag{16}$$

where $z = \exp(2\pi \mathrm{i} x)$, than by the numerical acceleration algorithm demonstrated by Oleksy. For this confluent hypergeometric function the Padé approximations are known in closed form (see [11, ch 13]). Thus, the [5, 5] Padé approximation with $a = 3/2$ gives

$$h = 0.487\,495\,52 + 0.266\,054\,04\mathrm{i}$$

where the figures are accurate to two digits in the last decimal place. Therefore

$$\sum_{n=0}^{\infty} \frac{\cos(n\pi/2)}{n+3/2} = 0.487\,495\,52 \qquad \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n+3/2} = 0.266\,0540. \tag{17}$$

However, for our present purposes the equivalent expansion in terms of Fourier transforms is of more interest, since as will be seen, the resulting series has convergence properties that are typical of what is expected for a wide class of functions. After straightforward manipulations (see appendix A), we find that application of equation (13) gives

$$\sum_{n=0}^{\infty} \frac{\cos 2\pi nx}{n+a} = \frac{1}{2a} + g(2\pi ax) + \sum_{n=1}^{\infty} [g(2\pi a(n+x)) + g(2\pi a(n-x))] \tag{18}$$

and

$$\sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{n+a} = \sum_{n=1}^{\infty} [f(2\pi a(n-x)) - f(2\pi a(n+x))] \tag{19}$$

for the real and imaginary parts of the exponential Fourier series, where the auxiliary functions f and g are defined by

$$g(y) \equiv \cos y\,\mathrm{Ci}(y) + \sin y\,\mathrm{Si}(y) \qquad f(y) \equiv \sin y\,\mathrm{Ci}(y) - \cos y\,\mathrm{Si}(y)$$

$$\mathrm{Si}(y) \equiv \int_y^{\infty} \frac{\sin t}{t}\,\mathrm{d}t \qquad \mathrm{Ci}(y) \equiv \int_y^{\infty} \frac{\cos t}{t}\,\mathrm{d}t.$$

Observing that the asymptotic formula for g is [12, p 233]

$$g(y) \sim \frac{1}{y^2}\left[1 - \frac{3!}{y^2} + \frac{5!}{y^4} - + \cdots\right] \tag{20}$$

it is clear that the transformed cosine series will converge much more rapidly than the original series. For example, if $a = 10$ and $x = 0.5$ (for which the sine series vanishes), the leading term becomes less than 0.000 001 at about $n = 40$. While this represents an enormous improvement over the original series, the convergence can be improved still further by application of the Kummer transformation [13, pp 171–2]. Adding and subtracting the first term in the asymptotic expansion

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\cos 2n\pi x}{n+a} &= \frac{1}{2a} + g(2\pi a x) + \sum_{n=1}^{\infty}\left[g(2\pi a(n+x)) - \frac{1}{(2\pi a)^2(n+x)^2}\right] \\ &\quad + \sum_{n=1}^{\infty}\left[g(2\pi a(n-x)) - \frac{1}{(2\pi a)^2(n+x)^2}\right] + \frac{\zeta(x,2)+\zeta(-x,2)}{(2\pi a)^2}\end{aligned} \tag{21}$$

where ζ is the Hurwitz zeta function [8, p 265] defined by

$$\zeta(x,s) \equiv \sum_{n=1}^{\infty} \frac{1}{(n+x)^s}$$

which can be easily evaluated, as described in appendix B. The difference between g and the first term in its asymptotic expansion tends to zero very rapidly, and all but the first four terms in the transformed series are less than 0.000 0001.

The transformed sine series may be similarly rearranged. Noting that the asymptotic expansion of f is

$$f(y) \sim \frac{1}{y}\left[1 - \frac{2!}{y^2} + \frac{4!}{y^4} - + \cdots\right] \tag{22}$$

the leading term may be added and subtracted as before, giving

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{n+a} &= \sum_{n=1}^{\infty}\left[f(2\pi a(n-x)) - \frac{1}{(2\pi a)(n-x)}\right] \\ &\quad - \sum_{n=1}^{\infty}\left[f(2\pi a(n+x)) - \frac{1}{(2\pi a)(n+x)}\right] + \frac{1}{4\pi a^2}\left[\frac{1}{\pi x} - \cot \pi x\right]\end{aligned} \tag{23}$$

where use has been made of the result

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - x^2} = \frac{\pi}{2x}\left[\frac{1}{\pi x} - \cot \pi x\right] \tag{24}$$

obtained by rearranging the Mittag–Leffler expansion of the cotangent function [10, p 175]. A particular advantage of the transformed series given by equations (21) and (23), as compared with both the numerical procedure described by Oleksy [2] and the representation as a hypergeometric series, is that the physically important limiting case when x tends to 0 or 1 (and the series is divergent) can be easily dealt with. In fact, the cosine series converges equally well for any other finite value of x, since the divergence is localized in the terms $g(2\pi a x)$ or $g(2\pi a(1-x))$, which are logarithmically singular as their respective arguments tend to zero. It is also important to point out that the need for the Kummer transformation to obtain equations (21) and (23) is a result of the asymptotic behaviour of the Fourier transforms that arise for this particular function. In many cases, the use of an additional transformation is unnecessary, as it is for some of the examples considered later in the paper.

4. Reciprocal quadratic coefficients

In the example considered in the previous section, both the real and imaginary parts of the exponential Fourier series had comparable convergence properties, determined by the sine and cosine integral functions. For the real and imaginary parts of

$$\sum_{n=0}^{\infty} \frac{\mathrm{e}^{\mathrm{i}cn}}{n^2+b^2} \equiv \sum_{n=0}^{\infty} \frac{\cos cn}{n^2+b^2} + \mathrm{i} \sum_{n=1}^{\infty} \frac{\sin cn}{n^2+b^2} \tag{25}$$

application of the appropriate forms of the PSF results in series with very different properties. The required Fourier sine and cosine transforms are readily available from the results

$$\int_0^{\infty} \frac{t-\mathrm{i}b}{t^2+b^2}\, \mathrm{e}^{\mathrm{i}ct}\, \mathrm{d}t = \mathrm{e}^{bc} \mathrm{E}_1(bc) \tag{26}$$

and

$$\int_0^{\infty} \frac{t+\mathrm{i}b}{t^2+b^2}\, \mathrm{e}^{\mathrm{i}ct}\, \mathrm{d}t = \mathrm{e}^{-bc}[-\mathrm{Ei}(bc) + \mathrm{i}\pi] \tag{27}$$

which are given by Abramowitz and Stegun [12, p 230], where E_1 and Ei are the exponential integral functions

$$\mathrm{E}_1(x) \equiv \int_x^{\infty} \frac{\mathrm{e}^{-t}}{t}\, \mathrm{d}t \qquad \mathrm{Ei}(x) \equiv \mathrm{PV}\int_{-\infty}^{x} \frac{\mathrm{e}^{t}}{t}\, \mathrm{d}t.$$

Addition of equations (26) and (27) and separation of the real and imaginary parts gives

$$\int_0^{\infty} \frac{t \cos ct}{t^2+b^2}\, \mathrm{d}t = \frac{1}{2}[\mathrm{e}^{bc}\mathrm{E}_1(bc) - \mathrm{e}^{-bc}\, \mathrm{Ei}(bc)] \tag{28}$$

and

$$\int_0^{\infty} \frac{t \sin ct}{t^2+b^2}\, \mathrm{d}t = \frac{\pi}{2}\, \mathrm{e}^{-bc} \tag{29}$$

and similarly, subtraction of equation (26) from equation (27) gives

$$\int_0^{\infty} \frac{b \sin ct}{t^2+b^2}\, \mathrm{d}t = \frac{1}{2}[\mathrm{e}^{bc}\mathrm{E}_1(bc) + \mathrm{e}^{-bc}\, \mathrm{Ei}(bc)] \tag{30}$$

and

$$\int_0^{\infty} \frac{b \cos ct}{t^2+b^2}\, \mathrm{d}t = \frac{\pi}{2}\, \mathrm{e}^{-bc}. \tag{31}$$

The real and imaginary parts of equation (13) are

$$\sum_{n=0}^{\infty} f(n) \cos cn = \frac{1}{2} f(0) + \int_0^{\infty} f(t) \cos ct\, \mathrm{d}t + \sum_{m=1}^{\infty} \left[\int_0^{\infty} f(t) \cos(2m\pi + c)t\, \mathrm{d}t + \int_0^{\infty} f(t) \cos(2m\pi - c)t\, \mathrm{d}t \right] \tag{32}$$

and

$$\sum_{n=1}^{\infty} f(n) \sin cn = \int_0^{\infty} f(t) \sin ct\, \mathrm{d}t + \sum_{m=1}^{\infty} \left[\int_0^{\infty} f(t) \sin(2m\pi + c)t\, \mathrm{d}t - \int_0^{\infty} f(t) \sin(2m\pi - c)t\, \mathrm{d}t \right]. \tag{33}$$

The real part of equation (25) is well known and can easily be derived by residue theory, as shown by Henrici [14, p 271]. The exponentials in the Fourier cosine transforms given by equation (31) can be summed as geometric series, giving, after transposing terms

$$\sum_{n=1}^{\infty}\frac{\cos cn}{b^2+n^2}=\frac{\pi}{2b}\cdot\frac{\cosh b(\pi-c)}{\sinh b\pi}-\frac{1}{2b^2}. \tag{34}$$

The imaginary part of equation (25) follows by use of equation (30) in equation (33)

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sin cn}{b^2+n^2}&=\frac{1}{2b}[\mathrm{e}^{bc}\mathrm{E}_1(bc)+\mathrm{e}^{-bc}\,\mathrm{Ei}(bc)]\\&\quad+\frac{1}{2b}\sum_{m=1}^{\infty}\{\mathrm{e}^{(2m\pi+c)b}\mathrm{E}_1[(2m\pi+c)b]+\mathrm{e}^{-(2m\pi+c)b}\,\mathrm{Ei}[(2m\pi+c)b]\\&\quad-\mathrm{e}^{(2m\pi-c)b}\mathrm{E}_1[(2m\pi-c)b]-\mathrm{e}^{-(2m\pi-c)b}\,\mathrm{Ei}[(2m\pi-c)b]\}\\&=\frac{1}{2b}[\mathrm{e}^{bc}\mathrm{E}_1(bc)+\mathrm{e}^{-bc}\,\mathrm{Ei}(bc)]\\&\quad+\sum_{m=1}^{\infty}\{\mathrm{e}^{(2m\pi+b)c}\mathrm{E}_1[(2m\pi+c)b]-\mathrm{e}^{(2m\pi-b)c}\mathrm{E}_1[(2m\pi-c)b]\}\\&\quad+\sum_{m=1}^{\infty}\{\mathrm{e}^{-(2m\pi+c)b}\,\mathrm{Ei}[(2m\pi+c)b]-\mathrm{e}^{-(2m\pi-c)b}\,\mathrm{Ei}[(2m\pi-c)b]\}.\end{aligned} \tag{35}$$

Noting that the asymptotic expansions of the exponential integral functions are

$$\mathrm{E}_1(x)\sim\frac{\mathrm{e}^{-x}}{x}\left[1-\frac{1!}{x}+\frac{2!}{x^2}-+\cdots\right]\qquad\mathrm{Ei}(x)\sim\frac{\mathrm{e}^{x}}{x}\left[1+\frac{1!}{x}+\frac{2!}{x^2}+\cdots\right] \tag{36}$$

it is seen that the terms in the transformed series decrease approximately as $1/(2m\pi\pm c)$. Further acceleration by the Kummer transformation is clearly required. However, it is clear from equations (36) that to get the same speed of convergence as obtained in the preceding section for the Oleksy series, two terms of the asymptotic expansion must be taken. Thus

$$\begin{aligned}&\sum_{m=1}^{\infty}\{\mathrm{e}^{(2m\pi+c)b}\mathrm{E}_1[(2m\pi+c)b]-\mathrm{e}^{(2m\pi-c)b}\mathrm{E}_1[(2m\pi-c)b]\}\\&\quad=\frac{1}{2b}\left[\cot\frac{c}{2}-\frac{2}{c}\right]-\frac{1}{4\pi^2b^2}\left[\zeta\left(\frac{c}{2\pi},2\right)-\zeta\left(\frac{-c}{2\pi},2\right)\right]\\&\qquad+\sum_{m=1}^{\infty}\left\{\mathrm{e}^{(2m\pi+c)b}\mathrm{E}_1[(2m\pi+c)b]-\mathrm{e}^{(2m\pi-c)b}\mathrm{E}_1[(2m\pi-c)b]\right.\\&\qquad\left.-\frac{1}{(2m\pi+c)b}+\frac{1}{(2m\pi+c)^2b^2}+\frac{1}{(2m\pi-c)b}-\frac{1}{(2m\pi-c)^2b^2}\right\}\end{aligned} \tag{37}$$

and similarly for the other series

$$\begin{aligned}&\sum_{m=1}^{\infty}\{\mathrm{e}^{-(2m\pi+c)b}\,\mathrm{Ei}[(2m\pi+c)b]-\mathrm{e}^{-(2m\pi-c)b}\,\mathrm{Ei}[(2m\pi-c)b]\}\\&\quad=\frac{1}{2b}\left[\cot\frac{c}{2}+\frac{2}{c}\right]-\frac{1}{4\pi^2b^2}\left[\zeta\left(\frac{c}{2\pi},2\right)-\zeta\left(\frac{-c}{2\pi},2\right)\right]\end{aligned}$$

$$+\sum_{m=1}^{\infty}\bigg\{\mathrm{e}^{-(2m\pi+c)b}\,\mathrm{Ei}[(2m\pi+c)b]-\mathrm{e}^{-(2m\pi-c)b}\,\mathrm{Ei}[(2m\pi-c)b]$$
$$-\frac{1}{(2m\pi+c)b}-\frac{1}{(2m\pi+c)^2b^2}+\frac{1}{(2m\pi-c)b}+\frac{1}{(2m\pi-c)^2b^2}\bigg\}. \tag{38}$$

A question that arises naturally from consideration of these examples and that of the previous section is that of how one can account for such very different convergence properties. In the next section, it is shown how these differences can be related to the power series expansions of the coefficient functions in the Fourier series.

5. Rational approximations to the coefficient functions

Sufficient conditions for the existence of a Fourier integral representation of a function $f(x)$ are that: (a) f is sectionally continuous in every finite interval of the x axis; (b) $f(x_0) = [f(x_0-0) + f(x_0+0)]/2$ at each point of discontinuity x_0; and (c) $|f(x)|$ is integrable from $-\infty$ to $+\infty$ [15, p 115]. For monotonically decreasing functions defined for positive values of x, to which attention is henceforth restricted, the absolute integrability condition in turn requires that $|f(x)| < M/x^{1+\alpha}$, for some positive constants M and α. It is, however, to be observed that integrals of the form

$$\int_0^{\infty} f(x)\cos tx\,\mathrm{d}x \qquad \int_0^{\infty} f(x)\sin tx\,\mathrm{d}x$$

can exist under weaker conditions, since the sub-integrals I_n between successive nodes of the circular functions have alternating signs, and by Leibniz' theorem, the sum of these integrals is convergent if $|I_n| \to 0$ as $n \to \infty$.

In general, an approximate calculation of the Fourier transforms of a function requires suitable approximations to the sum of its power series. Since finite Taylor polynomials have the limitation of being accurate only for small values of x, it is preferable to use Padé approximations to f (obtained, for example, from continued fraction developments) since these are well known [16] to be useful in determining the asymptotic behaviour of a function from the first few terms of the power series. Perhaps the most important fact for present purposes is that the convergence of a power series does not necessarily imply convergence of the corresponding continued fraction representation, and *vice versa*: it is well known that some continued fractions can converge even for power series with zero radius of convergence. More specifically, Henrici [17, pp 518–29] proves that the existence and uniqueness of a continued fraction development for a given power series follow only if the so-called Hankel determinants constructed from the coefficients of the power series do not vanish, and bases most of his analytical treatment on this hypothesis.

Let us, therefore, assume that the power series coefficients of f satisfy the conditions for existence and uniqueness of continued fraction development. Then the convergents of this continued fraction can be used to form a sequence of $[n, n+1]$ Padé approximants that for a given value of x represents the function arbitrarily closely. The coefficients in the numerators and denominators of these rational polynomials can be obtained in a variety of ways, of which one the simplest is the division algorithm described by Demidovich and Maron [18, pp 70–3]. Application of this algorithm to a function represented in the form

$$f(x) = \frac{c_{10} + c_{11}x + c_{12}x^2 + c_{13}x^3 + \cdots}{c_{00} + c_{01}x + c_{02}x^2 + c_{03}x^3 + \cdots} \tag{39}$$

results in the continued fraction

$$f(x) = \cfrac{c_{10}}{c_{00} + \cfrac{c_{20}x}{c_{10} + \cfrac{c_{30}x}{c_{20} + \cfrac{c_{40}x}{c_{30} + \cdots}}}} \tag{40}$$

where the denominators c are defined recursively by the equation

$$c_{jk} = \begin{vmatrix} c_{j-2,0} & c_{j-2,k+1} \\ c_{j-1,0} & c_{j-1,k+1} \end{vmatrix}. \tag{41}$$

Those convergents of the continued fraction (40) that multiply out to give $[n, n+1]$ Padé approximants tend to zero with increasing x, and can, therefore, be Fourier transformed. For example, if $c_{00} = 1$ and $c_{01} = c_{02} = \cdots = 0$, the $[0, 1]$ approximation

$$f(x) \approx \frac{c_{10}}{1 + (c_{20}x/c_{10})} \tag{42}$$

tends to zero with increasing x, and gives Fourier transforms

$$\int_0^\infty \frac{c_{10}}{1 + (c_{20}x/c_{10})} \cos tx \,\mathrm{d}x = (c_{10}^2/c_{20})\left[\cos\frac{c_{10}t}{c_{20}}\,\mathrm{Ci}\left(\frac{c_{10}t}{c_{20}}\right) + \sin\frac{c_{10}t}{c_{20}}\,\mathrm{Si}\left(\frac{c_{10}t}{c_{20}}\right)\right] \tag{43}$$

$$\int_0^\infty \frac{c_{10}}{1 + (c_{20}x/c_{10})} \sin tx \,\mathrm{d}x = (c_{10}^2/c_{20})\left[\cos\frac{c_{10}t}{c_{20}}\,\mathrm{Si}\left(\frac{c_{10}t}{c_{20}}\right) - \sin\frac{c_{10}t}{c_{20}}\,\mathrm{Ci}\left(\frac{c_{10}t}{c_{20}}\right)\right]. \tag{44}$$

Since the $[2, 3]$, $[4, 5]\ldots$ approximations can be broken down into partial fractions that contain reciprocal linear factors similar in form to equation (42), the asymptotic behaviour of the Fourier transforms of these functions can be expected to be similar. However, since the poles of these rational approximants do not necessarily lie on the real or imaginary axes, the Fourier transforms can be expected to contain Si and Ci functions of general complex argument.

By partial integration, the Fourier cosine transform of an even function can be expressed as the Fourier sine transform of an odd function (its derivative), if this integral exists. The Fourier cosine transform of an odd function can likewise be expressed as the Fourier sine transform of its even derivative. The qualitative behaviour of these two cases can be determined by supposing that instead of equation (39) the function f is represented by

$$f(x) = \frac{c_{10} + c_{11}x^2 + c_{12}x^4 + c_{13}x^6 + \cdots}{c_{00} + c_{01}x^2 + c_{02}x^4 + c_{03}x^6 + \cdots}. \tag{45}$$

Following the same procedure applied to equation (39), but with x in equations (40)–(42) replaced throughout by x^2, the expression for the first approximation to the Fourier cosine transform corresponding to equation (42) is

$$\int_0^\infty \frac{c_{10}}{1 + (c_{20}x^2/c_{10})} \cos tx \,\mathrm{d}x = \frac{c_{10}^2}{c_{20}} \frac{\pi}{2\sqrt{c_{10}/c_{20}}}\,\mathrm{e}^{-t\sqrt{c_{10}/c_{20}}} \tag{46}$$

and the first approximation to the Fourier sine transform corresponding to equation (43) is

$$\begin{aligned}\int_0^\infty &\frac{c_{10}}{1 + (c_{10}x^2/c_{20})} \sin tx \,\mathrm{d}x \\ &= \frac{c_{10}}{2\sqrt{c_{10}/c_{20}}}[\mathrm{e}^{t\sqrt{c_{10}/c_{20}}}\mathrm{E}_1(t\sqrt{c_{10}/c_{20}}) + \mathrm{e}^{-t\sqrt{c_{10}/c_{20}}}\,\mathrm{Ei}(t\sqrt{c_{10}/c_{20}})].\end{aligned} \tag{47}$$

The higher $[n, n+1]$ approximations are rational polynomials in x^2, whose poles lie in the upper half of the complex plane. Since these rational polynomials obey the conditions of

Jordan lemmas [4, pp 35–8] and can be expanded in reciprocal quadratic partial fractions, the Fourier cosine transforms can be found by the well known technique of integration around a semicircular contour in the upper half-plane. On the other hand, the Fourier sine transform will be approximated by a sum of exponential-integral functions of the form (47). The conclusion is that the Fourier cosine transforms will tend to zero exponentially with increasing t, while the Fourier sine transforms tend to zero as $1/t$.

If the function f has a power series consisting only of odd powers, it can be represented most generally in the form

$$f(x) = x\frac{c_{10} + c_{11}x^2 + c_{12}x^4 + c_{13}x^6 + \cdots}{c_{00} + c_{01}x^2 + c_{02}x^4 + c_{03}x^6 + \cdots} \tag{48}$$

so the first approximation to the Fourier cosine and sine transforms are

$$\int_0^\infty \frac{c_{10}}{1 + (c_{20}x^2/c_{10})} x\cos tx\,\mathrm{d}x = \frac{c_{10}}{2}[\mathrm{e}^{t\sqrt{c_{10}/c_{20}}}\mathrm{E}_1(t\sqrt{c_{10}/c_{20}}) - \mathrm{e}^{-t\sqrt{c_{10}/c_{20}}}\,\mathrm{Ei}(t\sqrt{c_{10}/c_{20}})] \tag{49}$$

from equation (28), and

$$\int_0^\infty \frac{c_{10}}{1 + (c_{20}x^2/c_{10})} x\sin tx\,\mathrm{d}x = \frac{c_{10}^2}{c_{20}}\frac{\pi}{2}\,\mathrm{e}^{-t\sqrt{c_{10}/c_{20}}} \tag{50}$$

from equation (29). For functions of this kind, the higher rational approximations are the same as those for even functions, but multiplied by an additional factor x. After resolution into partial quadratic fractions, the corresponding approximations to the Fourier cosine and sine transforms will consist of sums of functions of the kind given by equations (49) and (50), respectively. The convergence behaviour of transformed Fourier series with odd coefficient functions will be the opposite of those with even coefficient functions.

5.1. *Examples*

To make these ideas concerning the approximation of Fourier transforms more definite, let us consider the example $f(x) = \exp(-ax)$, where a is real and positive. For this function the Fourier transforms are particularly simple, viz

$$\int_0^\infty \mathrm{e}^{-ax}\sin tx\,\mathrm{d}x = \frac{t}{a^2 + t^2} \tag{51}$$

$$\int_0^\infty \mathrm{e}^{-ax}\cos tx\,\mathrm{d}x = \frac{a}{a^2 + t^2}. \tag{52}$$

Here, the appropriate Fourier series can be summed in closed form by identification as geometric series

$$\sum_{n=0}^\infty \mathrm{e}^{-na}\cos nc = \frac{1}{2}\left(\sum_{n=0}^\infty \mathrm{e}^{-n(a+\mathrm{i}c)} + \sum_{n=0}^\infty \mathrm{e}^{-n(a-\mathrm{i}c)}\right) = \frac{1 - \mathrm{e}^{-a}\cos c}{1 - 2\,\mathrm{e}^{-a}\cos c + \mathrm{e}^{-2a}} \tag{53}$$

$$\sum_{n=0}^\infty \mathrm{e}^{-na}\sin nc = \frac{1}{2\mathrm{i}}\left(\sum_{n=0}^\infty \mathrm{e}^{-n(a+\mathrm{i}c)} - \sum_{n=0}^\infty \mathrm{e}^{-n(a-\mathrm{i}c)}\right) = \frac{\mathrm{e}^{-a}\sin c}{1 - 2\,\mathrm{e}^{-a}\cos c + \mathrm{e}^{-2a}} \tag{54}$$

Schwatt [19, pp 211–43] gives a comprehensive discussion of these and other interesting examples of exactly summable trigonometric series. It would not, therefore, be necessary to accelerate the convergence by use of the PSF, but this example is still useful in illustrating the approximation of the Fourier transforms. From the continued fraction development, with

$c_{00} = 1$, $c_{01} = c_{02} = \cdots = 0$, $c_{10} = 1$, $c_{11} = -a$, and $c_{20} = a$, the $[0, 1]$ Padé approximation is $1/(1 + ax)$. The corresponding approximation to the Fourier cosine transform is

$$\int_0^\infty \frac{\cos tx}{1 + ax}\,\mathrm{d}x = \frac{1}{a}\int_0^\infty \frac{\cos tx}{1/a + x}\,\mathrm{d}x = \cos\frac{t}{a}\,\mathrm{Ci}\left(\frac{t}{a}\right) + \sin\frac{t}{a}\,\mathrm{Si}\left(\frac{t}{a}\right) \equiv g\left(\frac{t}{a}\right) \tag{55}$$

where g is the auxiliary function defined by equation (19), and taking into account the asymptotic expansion of g (equation (20))

$$g\left(\frac{t}{a}\right) \sim \left(\frac{a}{t}\right)^2 \tag{56}$$

it is clear that equation (55) provides an arbitrarily good approximation to the Fourier transform for sufficiently large t. Specifically, for some number $\varepsilon > 0$, the condition

$$\left|\frac{a}{t^2} - \frac{a}{a^2 + t^2}\right| < \varepsilon \tag{57}$$

is satisfied if $t > (a^3/\varepsilon)^{1/4}$. Clearly, similar procedures can be applied to the Fourier sine transform, but in this case the exact transform is asymptotic to $1/t$, which behaviour is also shown by the asymptotic expansion of the other auxiliary function given by equation (22).

The examples considered so far have involved functions possessing finitely many poles. As an example of a series in which the coefficients are even functions of n, and which has a countable infinity of poles, consider

$$\sum_{n=-\infty}^{\infty} \frac{\cos cn}{\cosh an} \tag{58}$$

which when a is close to zero converges slowly. According to Erdélyi *et al* [20, p 30] the required Fourier cosine transforms are all of the form

$$\int_{-\infty}^{\infty} \frac{\cos cx}{\cosh ax}\,\mathrm{d}x = \frac{\pi}{a\cosh(\pi c/2a)} \tag{59}$$

where c is replaced in the transformed series by $2\pi m \pm c$. The transformed series is therefore

$$\sum_{n=-\infty}^{\infty} \frac{\cos cn}{\cosh an} = \frac{\pi}{a}\sum_{m=-\infty}^{\infty} \frac{1}{\cosh(\pi/2a)(2m\pi + c)} \tag{60}$$

which converges extremely rapidly. The value of n required to make the absolute value of the coefficient function in the leading term of equation (58) less than ε is

$$n = \left[\frac{1}{a}\cosh^{-1}\frac{1}{\varepsilon}\right]. \tag{61}$$

For example, with $a = 0.1$, $c = 0.3$, and $\varepsilon = 10^{-7}$, the value of n required is 168. In contrast, the $m = 0$ term in equation (60) is $0.017\,965\,132(\pi/a)$, and the $m = 1$ term is $2.462\,170\,735 \times 10^{-45}(\pi/a)$!

The PSF can also be applied to series involving more complicated functions possessing branch points and similar singularities. The best known examples are the Madelung sums for coulomb potentials in ionic crystals, where the reciprocal distance between the lattice points involves a square root function, and the periodic alternation of signs of the ionic potentials can be regarded as a type of Fourier series. Many highly efficient and ingenious methods have been applied to the evaluation of lattice sums. The best known of these methods are based on various integral representations of the reciprocal distance, for example those involving Gaussian functions [21], or gamma functions [22]. Application of the PSF to the evaluation of coulomb potential lattice sums generally results in K_0 Bessel functions [23].

As another example of series of irrational (square root) terms, for which application of the PSF gives K_0 Bessel functions, let us consider the problem of developing a rapidly converging form of the series

$$G(x, y \mid \xi, \eta) = \frac{2}{a}\sum_{n=1}^{\infty} \frac{\sinh\sigma_n y \sinh\sigma_n(b-\eta)}{\sigma_n \sinh\sigma_n b} \sin\frac{n\pi x}{a} \sin\frac{n\pi\xi}{a} \qquad (y < \eta) \tag{62}$$

where

$$\sigma_n \equiv \sqrt{\beta^2 + \left(\frac{n\pi}{a}\right)^2}.$$

This can be identified (Roach [24, p 270]) as the Green function for the partial differential equation

$$\nabla^2 w - \beta^2 w = 0 \tag{63}$$

with Dirichlet boundary conditions, in a rectangular domain $0 < x < a$, $0 < y < b$. This equation arises in the solution of a linearized Navier–Stokes equation describing the development of steady laminar flow of an incompressible fluid in a rectangular duct. The parameter β is a function of z, which assumes large values near the mouth of the duct, but decays to zero as z increases to infinity, where (as is well known) the velocity profile is described by a series of rectangular harmonics. It is clear that in the physically important region where β is large, the series has poor convergence properties. Application of the PSF to

$$\sum_{n=-\infty}^{\infty} \frac{\sinh\sigma_n\eta \sinh\sigma_n(b-y)}{\sigma_n \sinh\sigma_n b}\, \mathrm{e}^{\mathrm{i}cn} \equiv \sum_{n=-\infty}^{\infty} f(n)\, \mathrm{e}^{\mathrm{i}cn} \tag{64}$$

requires evaluation of the integral

$$\int_{-\infty}^{\infty} f(t)\cos ct\, \mathrm{d}t \equiv \int_{-\infty}^{\infty} \frac{\sinh\sigma\eta \sinh\sigma(b-y)}{\sigma \sinh\sigma b} \cos ct\, \mathrm{d}t \tag{65}$$

where the subscript on σ has been dropped for clarity, remembering that σ is a function of t. The product of hyperbolic functions can be expanded as a geometric series of exponential terms by observing that

$$\frac{\sinh\sigma\eta \sinh\sigma(b-y)}{\sinh\sigma b} = \frac{\cosh\sigma(b-y+\eta) - \cosh\sigma(b-y-\eta)}{\sinh\sigma b} \tag{66}$$

and

$$\begin{aligned}\frac{\cosh\sigma(b-|y\pm\eta|)}{\sinh\sigma b} &= \frac{\mathrm{e}^{\sigma(b-|y\pm\eta|)} + \mathrm{e}^{-\sigma(b-|y\pm\eta|)}}{\mathrm{e}^{\sigma b} - \mathrm{e}^{-\sigma b}} \\ &= \sum_{k=0}^{\infty} \mathrm{e}^{-\sigma[2kb+|y\pm\eta|]} + \sum_{k=0}^{\infty} \mathrm{e}^{-\sigma[2(k+1)b+|y\pm\eta|]}.\end{aligned} \tag{67}$$

It is therefore sufficient to consider the function

$$G(p) = \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{-p\sqrt{\beta^2+(n\pi/a)^2}}}{\sqrt{\beta^2+(n\pi/a)^2}} \cos cn \tag{68}$$

where the parameter p depends linearly on y, η, b and k. If p is small and β is large, the magnitude of the exponential terms will be dominated by β and will tend to zero slowly with increasing n. The Fourier integral required for application of the PSF is therefore

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-p\sqrt{\beta^2+(x\pi/a)^2}}}{\sqrt{\beta^2+(x\pi/a)^2}} \cos cx\, \mathrm{d}x = \frac{a}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-(p\pi/a)\sqrt{x^2+(\beta a/\pi)^2}}}{\sqrt{x^2+(\beta a/\pi)^2}} \cos cx\, \mathrm{d}x. \tag{69}$$

This expression is not amenable to evaluation by residues, but can be evaluated according to the formula given by Erdélyi *et al* [20, p 17]

$$\int_0^\infty \frac{e^{-B\sqrt{x^2+A^2}}}{\sqrt{x^2+A^2}} \cos xt \, dx = K_0[A\sqrt{B^2+t^2}] \tag{70}$$

where K_0 is the modified Bessel function of the second kind, of order zero. This result may be derived as shown in appendix C from known Fourier and Laplace transforms. Making the identifications

$$A = \frac{\beta a}{\pi} \qquad B = \frac{p\pi}{a} \qquad t = c \tag{71}$$

the integral of interest is

$$\int_{-\infty}^\infty \frac{e^{-p\sqrt{\beta^2+(x\pi/a)^2}}}{\sqrt{\beta^2+(x\pi/a)^2}} \cos cx \, dx = \frac{2a}{\pi} K_0\left[\frac{a\beta}{\pi}\sqrt{\left(\frac{p\pi}{a}\right)^2 + c^2}\right]. \tag{72}$$

The transformed series is therefore

$$G(p) = \sum_{n=-\infty}^{\infty} \frac{e^{-p\sqrt{\beta^2+(n\pi/a)^2}}}{\sqrt{\beta^2+(n\pi/a)^2}} \cos cn = \frac{2a}{\pi} \sum_{m=-\infty}^{\infty} K_0\left[\frac{a\beta}{\pi}\sqrt{\left(\frac{p\pi}{a}\right)^2 + (2m\pi+c)^2}\right] \tag{73}$$

which converges extremely rapidly if β is large and p is small.

The PSF can be applied with equal success to more complicated cases where analytical expressions for the required transforms are not available. Purely numerical application of the PSF could, in principle, be based on any technique for evaluation of transforms (e.g. the fast Fourier transform algorithm). However, since the transform is typically required for only a few values of the transform variable, it is more efficient to determine the Fourier transforms by combination of a quadrature rule suited to integration between the nodes of the circular functions, and a sequence transformation applicable to sums of slowly decreasing terms of alternating signs (e.g. the Euler transformation). The optimal combination of these numerical techniques will be discussed in a subsequent paper. The analytical results presented here will facilitate the intelligent use of these numerical procedures by allowing identification of series for which use of the PSF is appropriate.

6. Conclusions

The purpose of this paper was to determine the conditions under which the PSF is useful in the transformation of slowly converging Fourier series. The extent of convergence acceleration achieved can be related to the asymptotic behaviour of the required Fourier transforms, and a method whereby this asymptotic behaviour could be determined from the power series of the coefficient functions was described.

For Fourier-transformable functions that can be approximation by continued fraction developments, approximations to the Fourier cosine and sine transforms can be obtained from the convergents that are equivalent to $(n, n+1)$ Padé approximants, after expansion in partial fractions. The pattern that emerges can be summarized as follows.

(1) For an even function $f(x)$, the Fourier cosine transform decreases exponentially with increasing transform variable t, but the Fourier sine transform involves exponential integral functions E_1 and Ei and tends to zero approximately as $1/t$.

(2) If $f(x)$ is odd, the Fourier sine transform decreases exponentially with increasing transform variable t, but the Fourier cosine transform involves exponential integral functions E_1 and Ei and tends to zero approximately as $1/t$.

(3) If $f(x)$ has a power series with both odd and even powers of x, the Fourier transforms can be approximated in terms of the sine and cosine integral functions Si and Ci, which are asymptotic to $1/t$ and $1/t^2$ respectively.

The PSF, therefore, works best for Fourier cosine series where the coefficient functions are even, and Fourier sine series where the coefficient functions are odd. In all other cases, application of the PSF produces a transformed series that, in general, requires additional convergence acceleration, for example, by the Kummer transformation.

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Disclaimer

Appendix A. Derivation of equations (18) and (19)

After making the substitution $u = t + a$, the first integral in the summand of equation (13) can be expressed in terms of the sine and cosine integral functions

$$\begin{aligned}\int_0^\infty \frac{\mathrm{e}^{2\pi\mathrm{i}(n+x)t}}{a+t}\,\mathrm{d}t &= \int_0^\infty \frac{\mathrm{e}^{2\pi\mathrm{i}(n+x)(u+a)}}{u}\,\mathrm{d}u = \mathrm{e}^{-2\pi a(n+x)\mathrm{i}}\int_{2\pi a(n+x)}^\infty \frac{\mathrm{e}^{\mathrm{i}t}}{t}\,\mathrm{d}t \\ &= [\cos 2\pi a(n+x) - \mathrm{i}\sin 2\pi a(n+x)][\mathrm{Ci}(2\pi a(n+x)) + \mathrm{i}\,\mathrm{Si}(2\pi a(n+x))] \\ &= \cos 2\pi a(n+x)\,\mathrm{Ci}(2\pi a(n+x)) + \sin 2\pi a(n+x)\,\mathrm{Si}(2\pi a(n+x)) \\ &\quad +\mathrm{i}[-\sin 2\pi a(n+x)\,\mathrm{Ci}(2\pi a(n+x)) + \cos 2\pi a(n+x)\,\mathrm{Si}(2\pi a(n+x))]\end{aligned} \tag{A.1}$$

where

$$\mathrm{Si}(y) \equiv \int_y^\infty \frac{\sin t}{t}\,\mathrm{d}t \qquad \mathrm{Ci}(y) \equiv \int_y^\infty \frac{\cos t}{t}\,\mathrm{d}t.$$

Proceeding similarly, the other integral is found to be

$$\begin{aligned}\int_0^\infty \frac{\mathrm{e}^{-2\pi\mathrm{i}(n-x)t}}{a+t}\,\mathrm{d}t \\ &= \cos 2\pi a(n-x)\,\mathrm{Ci}(2\pi a(n-x)) + \sin 2\pi a(n-x)\,\mathrm{Si}(2\pi a(n-x)) \\ &\quad +\mathrm{i}[\sin 2\pi a(n-x)\,\mathrm{Ci}(2\pi a(n-x)) - \cos 2\pi a(n-x)\,\mathrm{Si}(2\pi a(n-x))].\end{aligned} \tag{A.2}$$

The real and imaginary parts of the exponential Fourier series therefore are

$$\sum_{n=0}^\infty \frac{\cos 2\pi nx}{n+a} = \frac{1}{2a} + g(2\pi ax) + \sum_{n=1}^\infty [g(2\pi a(n+x)) + g(2\pi a(n-x))] \tag{A.3}$$

and

$$\sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{n+a} = \sum_{n=1}^{\infty} [f(2\pi a(n-x)) - f(2\pi a(n+x))] \tag{A.4}$$

where the auxiliary functions f and g are defined by

$$g(y) \equiv \cos y \,\mathrm{Ci}(y) + \sin y \,\mathrm{Si}(y) \qquad f(y) \equiv \sin y \,\mathrm{Ci}(y) - \cos y \,\mathrm{Si}(y).$$

In the present work, the sine and cosine integral functions were evaluated by summation of their power series expansions for small arguments (less than unity). For larger arguments, values of f and g were calculated by use of the rational polynomial approximants given by Abramowitz and Stegun [12, p 235] and the sine and cosine integrals were obtained by rearrangement of these definitions. Since the error limit for these polynomial approximants was stated to be about 5×10^{-7} for f and 3×10^{-7} for g, all calculations were carried out in single precision.

Appendix B. Evaluation of the Hurwitz zeta function

The series of functions defined by the equation

$$S_{p2} \equiv \sum_{n=1}^{\infty} \frac{1}{n^p (n+x)^2} \tag{B.1}$$

for $p = 0, 1, 2, \ldots$ satisfy a recurrence relation that can easily be established by application of the Kummer transformation

$$\begin{aligned} S_{p2} &= \sum_{n=1}^{\infty} \frac{1}{n^{p+2}} - \sum_{n=1}^{\infty} \left[\frac{1}{n^{p+2}} - \frac{1}{(n+x)^2 n^p} \right] \\ &= \sum_{n=1}^{\infty} \frac{1}{n^{p+2}} - \sum_{n=1}^{\infty} \frac{n^p (n+x)^2 - n^{p+2}}{n^{2p+2} (n+x)^2} \\ &= \zeta(p+2) - 2x S_{p+1,2} - x^2 S_{p+2,2} \end{aligned} \tag{B.2}$$

where $\zeta(p+2)$ is the Riemann zeta function of argument $p+2$. For $p = 3$ and $p = 4$, the series defined by equation (B.1) can be determined directly, and the recurrence used to obtain corresponding sums for $p = 2$, 1, and 0. After appropriate substitution and collecting terms the result is

$$\zeta(x, 2) \equiv S_{02} = \zeta(2) - 2x\xi(3) + 3x^2 \zeta(4) - 4x^3 S_{32} - 3x^4 S_{42}. \tag{B.3}$$

The required values of the Riemann zeta function are well known [12, p 811]

$$\zeta(2) = \frac{\pi^2}{6} \qquad \zeta(3) = 1.202\,0569\ldots \qquad \zeta(4) = \frac{\pi^4}{90}. \tag{B.4}$$

Appendix C

The exponential function may be represented as a Laplace transform by use of the formula [20, p 246]

$$\int_0^{\infty} \mathrm{e}^{-st} \frac{\mathrm{e}^{-a^2/4t}}{\sqrt{\pi t}} \,\mathrm{d}t = \frac{\mathrm{e}^{-a\sqrt{s}}}{\sqrt{s}} \tag{C.1}$$

so that

$$\int_0^\infty \frac{e^{-\beta\sqrt{x^2+a^2}}}{\sqrt{x^2+a^2}} \cos xy \, dx = \int_0^\infty \cos xy \left\{ \int_0^\infty \frac{e^{-t(x^2+a^2)-\beta^2/4t}}{\sqrt{\pi t}} \, dt \right\} dx$$
$$= \int_0^\infty \frac{e^{-ta^2-\beta^2/4t}}{\sqrt{\pi t}} \left\{ \int_0^\infty e^{-tx^2} \cos xy \, dx \right\} dt \tag{C.2}$$

on interchanging the order of integration. Evaluating the integral over x by use of the result [20, p 15]

$$\int_0^\infty e^{-tx^2} \cos xy \, dx = \frac{1}{2}\sqrt{\frac{\pi}{t}}\, e^{-y^2/4t}$$
$$\int_0^\infty \frac{e^{-ta^2-\beta^2/4t}}{\sqrt{\pi t}} \left\{ \int_0^\infty e^{-tx^2} \cos xy \, dx \right\} dt = \int_0^\infty \frac{e^{-ta^2-\beta^2/4t}}{\sqrt{\pi t}} \left\{ \frac{1}{2}\sqrt{\frac{\pi}{t}}\, e^{-y^2/4t} \right\} dt$$
$$= \frac{1}{2} \int_0^\infty \frac{1}{t}\, e^{-ta^2-(\beta^2+y^2)/4t} \, dt \tag{C.3}$$

which is recognizable as one of the integral representations of K_0 discussed by Watson [25, p 183–7].

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